Vertex-Coloring 2-Edge-Weighting of Graphs *

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Abstract

A *k-edge-weighting* w of a graph G is an assignment of an integer weight, $w(e) \in \{1, \ldots, k\}$, to each edge e. An edge weighting naturally induces a vertex coloring c by defining $c(u) = \sum_{u \sim e} w(e)$ for every $u \in V(G)$. A k-edge-weighting of a graph G is *vertex-coloring* if the induced coloring c is proper, i.e., $c(u) \neq c(v)$ for any edge $uv \in E(G)$.

Given a graph G and a vertex coloring c_0, does there exist an edge-weighting such that the induced vertex coloring is c_0? We investigate this problem by considering edge-weightings defined on an abelian group.

It was proved that every 3-colorable graph admits a vertex-coloring 3-edge-weighting [12]. Does every 2-colorable graph (i.e., bipartite graphs) admit a vertex-coloring 2-edge-weighting? We obtain several simple sufficient conditions for graphs to be vertex-coloring 2-edge-weighting. In particular, we show that 3-connected bipartite graphs admit vertex-coloring 2-edge-weighting.

Keywords: edge-weighting; vertex-coloring; 3-connected bipartite graph.

AMS subject classification (2000): 05C15.

1 Introduction

In this paper, we consider only finite, undirected and simple connected graphs. For a vertex v of a graph $G = (V, E)$, $N_G(v)$ denotes the set of vertices which are adjacent to v. If

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$v \in V(G)$ and $e \in E(G)$, we use $v \sim e$ to denote that v is an end-vertex of e, $\omega(G)$ denotes the number of connected components of G. An *k-vertex coloring* c of G is an assignment of k integers, $1, 2, \ldots, k$, to the vertices of G, the color of a vertex v is denoted by $c(v)$. The coloring is *proper* if no two distinct adjacent vertices share the same color. A graph G is *k-colorable* if G has a proper k-vertex coloring. The *chromatic number* $\chi(G)$ is the minimum number r such that G is r-colorable. Notation and terminology that is not defined here may be found in [6].

A *k-edge-weighting* w of a graph G is an assignment of an integer weight $w(e) \in \{1, \ldots, k\}$ to each edge e of G. An edge weighting naturally induces a vertex coloring $c(u)$ by defining $c(u) = \sum_{u \sim e} w(e)$ for every $u \in V(G)$. An k-edge-weighting of a graph G is *vertex-coloring* if for every edge $e = uv$, $c(u) \neq c(v)$ and then we say G admitting a *vertex-coloring k-edge-weighting.* **Moreover, we introduce a concept, which is different from the concept discussed here but similar enough. A multigraph is *irregular* if no two vertex degrees are equal. A multigraph can be viewed as a weighted graph with nonnegative-integer weights on the edges. The degree of a vertex in a weighted graph is the sum of the incident weights. Chartrand et al. [9] defined the *irregularity strength* of a graph G, written $s(G)$, to be the minimum of the maximum edge weight in an irregular multigraph with underlying graph G.**

If a graph has an edge as a component, clearly it can not have a vertex-coloring k-edge-weighting. So in this paper, we only consider graphs without K_2 component and refer such graphs as *nice graphs.*

In [12], Karoński, Łuczak and Thomason initiated the study of vertex-coloring k-edge-weighting and they brought forward a conjecture as following.

Conjecture 1.1 (1-2-3-Conjecture) *Every nice graph admits a vertex-coloring 3-edge-weighting.*

Furthermore, they proved that the conjecture holds for 3-colorable graphs (see Theorem 1 in [12]). For other graphs, Addario-Berry *et al.* [2] showed that every nice graph admits a vertex-coloring 30-edge-weighting. Addario-Berry, Dalal and Reed [3] improved the number of integers required to 16. Later, Wang and Yu [13] improved this bound to 13. Recently, Kalkowski, Karoński and Pfender [11] showed that every nice graph admits a vertex-coloring 5-edge-weighting, which is a great leap towards the 1-2-3-Conjecture.

In this paper, we focus on vertex-coloring 2-edge-weighting. In Section 2, we present several new results about vertex-coloring 2-edge-weighting.

Besides the existence problem of vertex-coloring k-edge-weighting, a natural question to ask is that given a graph G and a vertex coloring c_0, can we realize the coloring c_0 by a k-edge-weighting, i.e., does there exist an edge-weighting such that the induced vertex

coloring is c_0? For general graphs, it is not easy to find such an edge-weighting. However, if restricting edge weights to an abelian group, we obtain a neat positive answer for this even for a non-proper coloring c_0. In Section 3, we show that every 3-connected nice bipartite graph admits a vertex-coloring 2-edge-weighting.

2 Vertex-coloring 2-edge-weighting

For a graph G, there is a close relationship between 2-edge-weightings and graph factors. Namely, a 2-edge-weighting problem is equivalent to finding a special factor of graphs (see [2, 3]). So to find spanning subgraphs with pre-specified degree is an important part of edge-weighting. We shall use some of these results in our proofs.

Lemma 2.1 (Addario-Berry, Dalal and Reed, [3]) *Given a graph $G = (V, E)$, if for all $v \in V$, there are integers a_v^-, a_v^+ such that $a_v^- \leq \lfloor \frac{1}{2} d(v) \rfloor \leq a_v^+ < d(v)$, and*

$$a_v^+ \leq \min\{\frac{1}{2}(d(v) + a_v^-) + 1, 2(a_v^- + 1) + 1\},$$

then there exists a spanning subgraph H of G such that $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$.

Given an arbitrary vertex coloring c_0, we want to find an edge-weighting such that the induced vertex coloring is c_0? Under a weak condition, the next two theorems show that there exists an edge-weighting from an abelian group to $E(G)$ to induce c_0 for bipartite and non-bipartite graphs respectively.

Theorem 2.2 *Let G be a non-bipartite graph and $\Gamma = \{g_1, g_2, \ldots, g_k\}$ be a finite abelian group, where $k = |\Gamma|$. Let c_0 be any k-vertex coloring of G with color classes $\{U_1, \ldots, U_k\}$, where $|U_i| = n_i$ $(1 \leq i \leq k)$. If there exists an element $h \in \Gamma$ such that $n_1 g_1 + \cdots + n_k g_k = 2h$, then there is an edge-weighting with the elements of Γ such that the induced vertex coloring is c_0.*

Proof. Let c_0 be any k-vertex coloring with vertex partition $\{U_1, \ldots, U_k\}$, where every element in U_i is colored with g_i $(1 \leq i \leq k)$ such that $n_1 g_1 + \cdots + n_k g_k = 2h$.

Assign one edge with weight h and the rest with zero, so the sum of vertex colors is $2h$. We now adjust this initial weighting, while maintaining the sum of vertex weights, until all the vertices in U_i have color g_i $(1 \leq i \leq k)$. Suppose there exists a vertex $u \in U_i$ with the wrong color $g \neq g_i$. Since $n_1 g_1 + \cdots + n_k g_k = 2h$, there must be another vertex $v \in V(G)$ whose color is also wrong. Since G is non-bipartite, we can choose a walk of even length from u to v, which is always possible since $k \geq 3$. Traverse this walk, adding $g_i - g, g - g_i, g_i - g, \ldots$

alternately to the edges as they are encountered. This operation maintains the sum of vertex weights, leaves the colors of all but u and v unchanged, and yields one more vertex of correct color. Hence, repeated applications give the desired weighting. $\square$

Theorem 2.3 *Let G be a nice bipartite graph and $Z_2 = \{0, 1\}$. Let c_0 be any 2-vertex coloring of G with color classes $\{U_0, U_1\}$, where $|U_i| = n_i$ $(0 \leq i \leq 1)$* ***such that*** *$c_0(U_i) = i$,* ***for*** *$i = 0, 1$. If n_1 is even, then there exists an edge-weighting with the elements of Z_2 such that the induced vertex coloring is c_0.*

Proof. Let $g_1 = 0$ and $g_2 = 1$. If there is a vertex u of color g_i with the wrong color $g \neq g_i$ and since n_2 is even, then there must be another vertex $v \in V(G)$ whose color is also wrong. Since G is connected, then there is a path from u to v. **Traverse this walk, adding $1, 1, 1, \ldots$ to the edges as they are encountered.** This operation always maintains the sum of vertex colors, leaves the colors of all but u and v unchanged, and yields one more vertex of correct weight. $\square$

Note that in Theorem 2.2, the given vertex-coloring c_0 can be either a proper or an improper coloring.

Remark: The edge-weighting problem on groups has been studied by Karoński, Łuczak and Thomason in [12]. They proved that for each $|\Gamma|$-colorable graph G, there exists an edge-weighting with the elements of Γ such that the induced vertex-coloring is proper. Our proof of Theorems 2.2 and 2.3 are modifications of the result.

It was proved in [12] that every 3-colorable graph has a vertex-coloring 3-edge-weighting. A natural question to ask is that whether every 2-colorable graph has a vertex-coloring 2-edge-weighting. In [8], Chang *et al.* considered vertex-coloring 2-edge-weighting in bipartite graphs and proved the following results.

Lemma 2.4 (Chang, Lu, Wu and Yu, [8])

Every connected nice bipartite graph admits a vertex-coloring 2-edge-weighting if one of following conditions holds:

(1) *$|A|$ or $|B|$ is even;*

(2) *$\delta(G) = 1$;*

(3) *$\lfloor d(u)/2 \rfloor + 1 \neq d(v)$ for any edge $uv \in E(G)$.*

Theorem 2.5 *Let G be a nice graph. If $\delta(G) \geq 8\chi(G)$, then G admits a vertex-coloring 2-edge-weighting.*

Proof. Let $\{V_1, \ldots, V_{\chi(G)}\}$ be a partition of $V(G)$ into independent sets. For each $v \in V_i$, choose a_v^- such that $\lfloor \frac{d(v)}{4} \rfloor \leq a_v^- \leq \lfloor \frac{d(v)}{2} \rfloor$, $a_v^- + d_G(v) \equiv 2i \pmod{2\chi(G)}$, and $a_v^- + 2\chi(G) \geq \lfloor \frac{d(v)}{2} \rfloor$. Such choice for a_v^- exists as $\delta(G) \geq 8\chi(G)$. Set $a_v^+ = a_v^- + 2\chi(G)$.

Furthermore, such a choices of a_v^- and a_v^+ satisfy the conditions of Lemma 2.1, i.e.,

$$\begin{aligned} \frac{1}{2}(d(v) - a_v^- - 2\chi(G)) - \chi(G) &= \frac{1}{2}(d(v) - a_v^+) - \chi(G) \\ &\geq \frac{d(v)}{8} - \chi(G), \end{aligned}$$

thus there is a subgraph H such that for all v, $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$. Set $w(e) = 2$ for $e \in E(H)$ and $w(e) = 1$ for $e \in E(G) - E(H)$. If $v \in V_i$, we have

$$\sum_{v \sim e} w(e) = 2d_H(v) + d_{G-H}(v) = d_G(v) + d_H(v) \in \{2i, 2i+1\} \pmod{2\chi(G)}.$$

Thus adjacent vertices in different parts of $\{V_1, \ldots, V_{\chi(G)}\}$ have different arities. As each V_i is an independent set, these weights form a vertex-coloring 2-edge-weighting of G. □

Theorem 2.6 *Given a nice bipartite graph $G = (U, W)$, if there exists a vertex v such that $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ and $G - v - N(v)$ is connected, then G admits a vertex-coloring 2-edge-weighting.*

Proof. If $|U| \cdot |W|$ is even, by Lemma 2.4, the result follows. So we may assume that both $|U|$ and $|W|$ are odd. Let $v \in U$ satisfy $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ and $N(v) = \{w_1, \ldots, w_k\}$. Since $G - v - N(v)$ is connected, by Theorem 2.3, $G - v - N(v)$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in U - v$ and $c(y)$ is even for all $y \in W - N(v)$. Now we assign every edge of $E[N(v), U]$ with weight 2. Clearly $c(x)$ is odd for all $x \in U - v$ and $c(y)$ is even for all $y \in W$. Moreover $c(v) \neq c(u)$ for all $u \in N(v)$ since $d(u) \neq d(v)$. Thus we obtain a vertex-coloring 2-edge-weighting of G. □

Theorem 2.7 *Given a nice bipartite graph $G = (U, W)$, if there exists a vertex v of degree $\delta(G)$ such that $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ and $G - v$ is connected, then G admits a vertex-coloring 2-edge-weighting.*

Proof. If $|U| \cdot |W|$ is even, by Lemma 2.4, the result follows. So we may assume that both $|U|$ and $|W|$ are odd. Let $v \in U$ satisfy $d_G(v) = \delta(G)$ and $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$. Now we consider two cases.

Case 1. $\delta(G)$ is even.

In this case, $|(U - v) \cup N(v)|$ is even and $W - N(v)$ is odd. By Theorem 2.3, $G - v$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in (U - v) \cup N(v)$ and

$c(y)$ is even for all $y \in W - N(v)$. Since $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$, assigning the edges incident to v with weight 1 induces a vertex-coloring 2-edge-weighting of G.

Case 2. $\delta(G)$ is odd.

In this case, $|(U - v) \cup N(v)|$ is odd and $W - N(v)$ is even. By Theorem 2.3, $G - v$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is even for all $x \in (U - v) \cup N(x)$ and $c(y)$ is odd for all $y \in W - N(v)$. Since $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$, assigning the edges incident to v with weight 1 induces a vertex-coloring 2-edge-weighting of G. □

3 3-connected bipartite graphs

An interesting corollary of Lemma 2.4 is that every *r-regular* nice bipartite graph ($r \geq 3$) admits a vertex-coloring 2-edge-weighting. **Note $C_6, C_{10}, \ldots$ are 2-regular nice bipartite graphs which do not admit vertex-coloring 2-edge-weightings.**

In the following, we continue the research in this direction and prove that a vertex-coloring 2-edge-weighting exists for every 3-connected bipartite graph. The following lemma is an important step in proving our main result.

Lemma 3.1 *Let G be a 3-connected non-regular bipartite graph with bipartition (U, W). Let $u \in U$ with $d(u) = \delta(G)$ and $t \leq \delta - 1$. Denote $N^\delta(u) = \{v \mid d(v) = \delta, v \in N_G(u)\} = \{u_1, \ldots, u_t\}$. Then there exist $e_1, \ldots, e_t$, where e_i is incident to vertex u_i in $G - u$ for $i = 1, \ldots, t$, such that $G - u - \{e_1, \ldots, e_t\}$ is connected.*

Proof. Let $C_1, \ldots, C_s$ be the components of $G - u - N^\delta(u)$. We construct a bipartite multi-graph H with bipartition (X, Y), where $X = \{u_1 \ldots, u_t\}$, $Y = \{c_1, \ldots, c_s\}$ and $|E_H(u_i, c_j)| = |E_G(u_i, C_j)|$ for $1 \leq i \leq t$ and $1 \leq j \leq s$. Then $d_H(u_i) = \delta - 1$ for every $u_i \in X$.

Claim. H contains a connected spanning subgraph T such that $d_T(v) \leq \delta - 2$ for every $v \in X$.

Suppose that the claim does not hold. Let R be a connected induced subgraph of H satisfying

i). R contains a connected spanning subgraph M such that $d_M(v) \leq \delta - 2$ for every $v \in V(M) \cap X$;

ii). $|V(R)|$ is maximum.

It is easy to see that $V(R) \neq \emptyset$ and $R \neq H$. Let $R = (A, B)$, where $A \subseteq X$ and $B \subseteq Y$. By maximality of R, we have $d_R(v) \geq \delta - 2$ for every $v \in A$ and $E_H(B, X - A) = \emptyset$. Let $L = \{v \mid d_R(v) = \delta - 2, v \in A\}$. We see $|L| \geq 2$ since G is 3-connected. Let M^* be a

connected spanning subgraph of R such that $d_{M^*}(v) = \delta - 2$ for every $v \in A$. Note that for every connected spanning subgraph N^* of M^*, we have $d_{N^*}(w) = \delta - 2$ for $w \in L$ by the maximality of R. So every edge incident with w in M^*, where $w \in L$, is a cut-edge of M^*. Let $|L| = l$ and $|E(R) - E(M^*)| = m$. Then $l + m \leq t \leq \delta - 1$. We have

$$\omega(M^* - L) = \omega(H - L - (E(R) - E(M^*))) - 1 \geq (\delta - 3)l + 1.$$

So $\omega(H - L) \geq (\delta - 3)l + 2 - m$, which implies

$$\begin{aligned}\omega(G - u - L) &\geq (\delta - 3)l + 1 - m + 1\\ &\geq (\delta - 3)l + 2 - (\delta - 1 - l)\\ &= (\delta - 2)l + 3 - \delta.\end{aligned}$$

Since G is 3-connected, then

$$3\omega(G - u - L) \leq (\delta - 1)l + \delta - l.$$

It follows that

$$\omega(G - u - L) \leq \lfloor \frac{(\delta - 1)l + \delta - l}{3} \rfloor.$$

However

$$(\delta - 2)l + 3 - \delta - \frac{(\delta - 1)l + \delta - l}{3} = \frac{2\delta l}{3} - \frac{4\delta}{3} - \frac{4l}{3} + 3 > 0,$$

a contradiction. So we complete the claim and thus obtain a connected spanning subgraph T of H.

Let E' denote the set of corresponding edges of $E(T)$ in G. Then we obtain a spanning subgraph $T^* = \bigcup_{i=1}^{s} C_i \cup N^\delta(u) \cup E'$ of $G - u$ such that $d_{T^*}(v) \leq \delta - 2$ for every $v \in N^\delta(u)$. Thus the proof is complete. □

The following theorem is the main result of this section.

Theorem 3.2 *Let $G = (U, W)$ be a nice bipartite graph. If G is 3-connected, then G admits a vertex-coloring 2-edge-weighting.*

Proof. If G is a regular graph, the result follows from Lemma 2.4 (3). In the following, let G be a 3-connected non-regular bipartite graph with bipartition (U, W). Let $u \in U$ with $d(u) = \delta(G)$ and $N^\delta(u) = \{v \mid d(v) = \delta, v \in N_G(u)\} = \{u_1, \ldots, u_t\}$, where $t \leq \delta - 1$. Then by Lemma 3.1, there exist $e_1, \ldots, e_t$, where e_i is incident to vertex u_i in $G - u$ for $i = 1, \ldots, t$, such that $G - u - \{e_1, \ldots, e_t\}$ is connected.

By Lemma 2.4, we can assume that $|U||W|$ is odd. Now we consider two cases.

Case 1. $\delta(G)$ is even.

Then $|N(u) \cup (U-u)|$ is even. By Theorem 2.3, $G-u-\{e_1,\ldots,e_t\}$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in N(u) \cup (U-u)$ and $c(y)$ is even for all $y \in W - N(u)$. We assign every edge of $\{e_1,\ldots,e_t\}$ with weight 2 and every edge of $\{uu_i \mid i = 1,\ldots,t\}$ with weight 1. If $d(u_i) = d(u)$ for $i = 1,\ldots,t$, then $d_{G-u-\{e_1,\ldots,e_t\}}(u_i)$ is even. Now $c(u) = d(u)$ and $c(u) < c(u_i)$ for $i = 1,\ldots,t$. Moreover, $c(u_i)$ is even for $i = 1,\ldots,t$. Hence we obtain a vertex-coloring 2-edge-weighting of G.

Case 2. $\delta(G)$ is odd.

Then $|W - N(u)|$ is even. By Theorem 2.3, $G-u-\{e_1,\ldots,e_t\}$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is even for all $x \in N(u) \cup (U-u)$ and $c(y)$ is odd for all $y \in W - N(u)$. We again assign every edge of $\{e_1,\ldots,e_t\}$ with weight 2 and every edge of $\{uu_i \mid i = 1,\ldots,t\}$ with weight 1. Then $c(u) = d(u)$ and $c(u) < c(u_i)$ for $i = 1,\ldots,t$. Moreover, $c(u_i)$ is odd for $i = 1,\ldots,t$. Then we obtain a vertex-coloring 2-edge-weighting of G.

We complete the proof. □

Based on the proof of Theorem 3.2, we can easily obtain the following corollary.

Corollary 3.3 *Let $G = (U,W)$ be a bipartite graph with $\delta(G) \geq 3$. If there exists a vertex of degree $\delta(G)$ such that $G-u-N(u)$ is connected, then G admits a vertex-coloring 2-edge-weighting.*

4 Conclusions

In this paper, we prove that every 3-connected bipartite graph has a vertex-coloring 2-edge-weighting. There exists a family of infinite bipartite graphs (e.g., the generalized θ-graphs) which is 2-connected and has a vertex-coloring 3-edge-weighting but not a vertex-coloring 2-edge-weighting. It remains an open problem to classify all 2-connected bipartite graphs admitting a vertex-coloring 2-edge-weighting.

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$v \in V(G)$ and $e \in E(G)$, we use $v \sim e$ to denote that v is an end-vertex of e, $\omega(G)$ denotes the number of connected components of G. An *k-vertex coloring* c of G is an assignment of k integers, $1, 2, \ldots, k$, to the vertices of G, the color of a vertex v is denoted by $c(v)$. The coloring is *proper* if no two distinct adjacent vertices share the same color. A graph G is *k-colorable* if G has a proper k-vertex coloring. The *chromatic number* $\chi(G)$ is the minimum number r such that G is r-colorable. Notation and terminology that is not defined here may be found in [6].

A *k-edge-weighting* w of a graph G is an assignment of an integer weight $w(e) \in \{1, \ldots, k\}$ to each edge e of G. An edge weighting naturally induces a vertex coloring $c(u)$ by defining $c(u) = \sum_{u \sim e} w(e)$ for every $u \in V(G)$. An k-edge-weighting of a graph G is *vertex-coloring* if for every edge $e = uv$, $c(u) \neq c(v)$ and then we say G admitting a *vertex-coloring k-edge-weighting*. Moreover, we introduce a concept, which is different from the concept discussed here but similar enough. A multigraph is *irregular* if no two vertex degrees are equal. A multigraph can be viewed as a weighted graph with nonnegative-integer weights on the edges. The degree of a vertex in a weighted graph is the sum of the incident weights. Chartrand et al. [9] defined the *irregularity strength* of a graph G, written $s(G)$, to be the minimum of the maximum edge weight in an irregular multigraph with underlying graph G.

If a graph has an edge as a component, clearly it can not have a vertex-coloring k-edge-weighting. So in this paper, we only consider graphs without K_2 component and refer such graphs as *nice graphs*.

In [12], Karoński, Łuczak and Thomason initiated the study of vertex-coloring k-edge-weighting and they brought forward a conjecture as following.

Conjecture 1.1 (1-2-3-Conjecture) *Every nice graph admits a vertex-coloring 3-edge-weighting.*

Furthermore, they proved that the conjecture holds for 3-colorable graphs (see Theorem 1 in [12]). For other graphs, Addario-Berry *et al.* [2] showed that every nice graph admits a vertex-coloring 30-edge-weighting. Addario-Berry, Dalal and Reed [3] improved the number of integers required to 16. Later, Wang and Yu [13] improved this bound to 13. Recently, Kalkowski, Karoński and Pfender [11] showed that every nice graph admits a vertex-coloring 5-edge-weighting, which is a great leap towards the 1-2-3-Conjecture.

In this paper, we focus on vertex-coloring 2-edge-weighting. In Section 2, we present several new results about vertex-coloring 2-edge-weighting.

Besides the existence problem of vertex-coloring k-edge-weighting, a natural question to ask is that given a graph G and a vertex coloring c_0, can we realize the coloring c_0 by a k-edge-weighting, i.e., does there exist an edge-weighting such that the induced vertex

coloring is c_0? For general graphs, it is not easy to find such an edge-weighting. However, if restricting edge weights to an abelian group, we obtain a neat positive answer for this even for a non-proper coloring c_0. In Section 3, we show that every 3-connected nice bipartite graph admits a vertex-coloring 2-edge-weighting.

2 Vertex-coloring 2-edge-weighting

For a graph G, there is a close relationship between 2-edge-weightings and graph factors. Namely, a 2-edge-weighting problem is equivalent to finding a special factor of graphs (see [2, 3]). So to find spanning subgraphs with pre-specified degree is an important part of edge-weighting. We shall use some of these results in our proofs.

Lemma 2.1 (Addario-Berry, Dalal and Reed, [3]) *Given a graph $G = (V, E)$, if for all $v \in V$, there are integers a_v^-, a_v^+ such that $a_v^- \leq \lfloor \frac{1}{2}d(v) \rfloor \leq a_v^+ < d(v)$, and*

$$a_v^+ \leq \min\{\frac{1}{2}(d(v) + a_v^-) + 1, 2(a_v^- + 1) + 1\},$$

then there exists a spanning subgraph H of G such that $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$.

Given an arbitrary vertex coloring c_0, we want to find an edge-weighting such that the induced vertex coloring is c_0? Under a weak condition, the next two theorems show that there exists an edge-weighting from an abelian group to $E(G)$ to induce c_0 for bipartite and non-bipartite graphs respectively.

Theorem 2.2 *Let G be a non-bipartite graph and $\Gamma = \{g_1, g_2, \ldots, g_k\}$ be a finite abelian group, where $k = |\Gamma|$. Let c_0 be any k-vertex coloring of G with color classes $\{U_1, \ldots, U_k\}$, where $|U_i| = n_i$ $(1 \leq i \leq k)$. If there exists an element $h \in \Gamma$ such that $n_1g_1 + \cdots + n_kg_k = 2h$, then there is an edge-weighting with the elements of Γ such that the induced vertex coloring is c_0.*

Proof. Let c_0 be any k-vertex coloring with vertex partition $\{U_1, \ldots, U_k\}$, where every element in U_i is colored with g_i $(1 \leq i \leq k)$ such that $n_1g_1 + \cdots + n_kg_k = 2h$.

Assign one edge with weight h and the rest with zero, so the sum of vertex colors is $2h$. We now adjust this initial weighting, while maintaining the sum of vertex weights, until all the vertices in U_i have color g_i $(1 \leq i \leq k)$. Suppose there exists a vertex $u \in U_i$ with the wrong color $g \neq g_i$. Since $n_1g_1 + \cdots + n_kg_k = 2h$, there must be another vertex $v \in V(G)$ whose color is also wrong. Since G is non-bipartite, we can choose a walk of even length from u to v, which is always possible since $k \geq 3$. Traverse this walk, adding $g_i - g, g - g_i, g_i - g, \ldots$

alternately to the edges as they are encountered. This operation maintains the sum of vertex weights, leaves the colors of all but u and v unchanged, and yields one more vertex of correct color. Hence, repeated applications give the desired weighting. $\square$

Theorem 2.3 *Let G be a nice bipartite graph and $Z_2 = \{0, 1\}$. Let c_0 be any 2-vertex coloring of G with color classes $\{U_0, U_1\}$, where $|U_i| = n_i$ $(0 \le i \le 1)$ such that $c_0(U_i) = i$, for $i = 0, 1$. If n_1 is even, then there exists an edge-weighting with the elements of Z_2 such that the induced vertex coloring is c_0.*

Proof. Let $g_1 = 0$ and $g_2 = 1$. If there is a vertex u of color g_i with the wrong color $g \neq g_i$ and since n_2 is even, then there must be another vertex $v \in V(G)$ whose color is also wrong. Since G is connected, then there is a path from u to v. Traverse this walk, adding $1, 1, 1, \ldots$ to the edges as they are encountered. This operation always maintains the sum of vertex colors, leaves the colors of all but u and v unchanged, and yields one more vertex of correct weight. $\square$

Note that in Theorem 2.2, the given vertex-coloring c_0 can be either a proper or an improper coloring.

Remark: The edge-weighting problem on groups has been studied by Karoński, Łuczak and Thomason in [12]. They proved that for each $|\Gamma|$-colorable graph G, there exists an edge-weighting with the elements of Γ such that the induced vertex-coloring is proper. Our proof of Theorems 2.2 and 2.3 are modifications of the result.

It was proved in [12] that every 3-colorable graph has a vertex-coloring 3-edge-weighting. A natural question to ask is that whether every 2-colorable graph has a vertex-coloring 2-edge-weighting. In [8], Chang *et al.* considered vertex-coloring 2-edge-weighting in bipartite graphs and proved the following results.

Lemma 2.4 (Chang, Lu, Wu and Yu, [8])

Every connected nice bipartite graph admits a vertex-coloring 2-edge-weighting if one of following conditions holds:

(1) $|A|$ *or* $|B|$ *is even;*

(2) $\delta(G) = 1$;

(3) $\lfloor d(u)/2 \rfloor + 1 \neq d(v)$ *for any edge* $uv \in E(G)$.

Theorem 2.5 *Let G be a nice graph. If $\delta(G) \geq 8\chi(G)$, then G admits a vertex-coloring 2-edge-weighting.*

Proof. Let $\{V_1, \ldots, V_{\chi(G)}\}$ be a partition of $V(G)$ into independent sets. For each $v \in V_i$, choose a_v^- such that $\lfloor \frac{d(v)}{4} \rfloor \le a_v^- \le \lfloor \frac{d(v)}{2} \rfloor$, $a_v^- + d_G(v) \equiv 2i \pmod{2\chi(G)}$, and $a_v^- + 2\chi(G) \ge \lfloor \frac{d(v)}{2} \rfloor$. Such choice for a_v^- exists as $\delta(G) \ge 8\chi(G)$. Set $a_v^+ = a_v^- + 2\chi(G)$.

Furthermore, such a choices of a_v^- and a_v^+ satisfy the conditions of Lemma 2.1, i.e.,

$$\begin{aligned}\frac{1}{2}(d(v) - a_v^- - 2\chi(G)) - \chi(G) &= \frac{1}{2}(d(v) - a_v^+) - \chi(G) \\ &\ge \frac{d(v)}{8} - \chi(G),\end{aligned}$$

thus there is a subgraph H such that for all v, $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$. Set $w(e) = 2$ for $e \in E(H)$ and $w(e) = 1$ for $e \in E(G) - E(H)$. If $v \in V_i$, we have

$$\sum_{v \sim e} w(e) = 2d_H(v) + d_{G-H}(v) = d_G(v) + d_H(v) \in \{2i, 2i+1\} \pmod{2\chi(G)}.$$

Thus adjacent vertices in different parts of $\{V_1, \ldots, V_{\chi(G)}\}$ have different arities. As each V_i is an independent set, these weights form a vertex-coloring 2-edge-weighting of G. $\square$

Theorem 2.6 *Given a nice bipartite graph $G = (U, W)$, if there exists a vertex v such that $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ and $G - v - N(v)$ is connected, then G admits a vertex-coloring 2-edge-weighting.*

Proof. If $|U| \cdot |W|$ is even, by Lemma 2.4, the result follows. So we may assume that both $|U|$ and $|W|$ are odd. Let $v \in U$ satisfy $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ and $N(v) = \{w_1, \ldots, w_k\}$. Since $G - v - N(v)$ is connected, by Theorem 2.3, $G - v - N(v)$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in U - v$ and $c(y)$ is even for all $y \in W - N(v)$. Now we assign every edge of $E[N(v), U]$ with weight 2. Clearly $c(x)$ is odd for all $x \in U - v$ and $c(y)$ is even for all $y \in W$. Moreover $c(v) \ne c(u)$ for all $u \in N(v)$ since $d(u) \ne d(v)$. Thus we obtain a vertex-coloring 2-edge-weighting of G. $\square$

Theorem 2.7 *Given a nice bipartite graph $G = (U, W)$, if there exists a vertex v of degree $\delta(G)$ such that $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ and $G - v$ is connected, then G admits a vertex-coloring 2-edge-weighting.*

Proof. If $|U| \cdot |W|$ is even, by Lemma 2.4, the result follows. So we may assume that both $|U|$ and $|W|$ are odd. Let $v \in U$ satisfy $d_G(v) = \delta(G)$ and $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$. Now we consider two cases.

Case 1. $\delta(G)$ is even.

In this case, $|(U - v) \cup N(v)|$ is even and $W - N(v)$ is odd. By Theorem 2.3, $G - v$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in (U - v) \cup N(v)$ and

$c(y)$ is even for all $y \in W - N(v)$. Since $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$, assigning the edges incident to v with weight 1 induces a vertex-coloring 2-edge-weighting of G.

Case 2. $\delta(G)$ is odd.

In this case, $|(U - v) \cup N(v)|$ is odd and $W - N(v)$ is even. By Theorem 2.3, $G - v$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is even for all $x \in (U - v) \cup N(x)$ and $c(y)$ is odd for all $y \in W - N(v)$. Since $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$, assigning the edges incident to v with weight 1 induces a vertex-coloring 2-edge-weighting of G. □

3 3-connected bipartite graphs

An interesting corollary of Lemma 2.4 is that every *r-regular* nice bipartite graph ($r \geq 3$) admits a vertex-coloring 2-edge-weighting. Note $C_6, C_{10}, \ldots$ are 2-regular nice bipartite graphs which do not admit vertex-coloring 2-edge-weightings.

In the following, we continue the research in this direction and prove that a vertex-coloring 2-edge-weighting exists for every 3-connected bipartite graph. The following lemma is an important step in proving our main result.

Lemma 3.1 *Let G be a 3-connected non-regular bipartite graph with bipartition (U, W). Let $u \in U$ with $d(u) = \delta(G)$ and $t \leq \delta - 1$. Denote $N^\delta(u) = \{v \mid d(v) = \delta, v \in N_G(u)\} = \{u_1, \ldots, u_t\}$. Then there exist $e_1, \ldots, e_t$, where e_i is incident to vertex u_i in $G - u$ for $i = 1, \ldots, t$, such that $G - u - \{e_1, \ldots, e_t\}$ is connected.*

Proof. Let $C_1, \ldots, C_s$ be the components of $G - u - N^\delta(u)$. We construct a bipartite multi-graph H with bipartition (X, Y), where $X = \{u_1 \ldots, u_t\}$, $Y = \{c_1, \ldots, c_s\}$ and $|E_H(u_i, c_j)| = |E_G(u_i, C_j)|$ for $1 \leq i \leq t$ and $1 \leq j \leq s$. Then $d_H(u_i) = \delta - 1$ for every $u_i \in X$.

Claim. H contains a connected spanning subgraph T such that $d_T(v) \leq \delta - 2$ for every $v \in X$.

Suppose that the claim does not hold. Let R be a connected induced subgraph of H satisfying

i). R contains a connected spanning subgraph M such that $d_M(v) \leq \delta - 2$ for every $v \in V(M) \cap X$;

ii). $|V(R)|$ is maximum.

It is easy to see that $V(R) \neq \emptyset$ and $R \neq H$. Let $R = (A, B)$, where $A \subseteq X$ and $B \subseteq Y$. By maximality of R, we have $d_R(v) \geq \delta - 2$ for every $v \in A$ and $E_H(B, X - A) = \emptyset$. Let $L = \{v \mid d_R(v) = \delta - 2, v \in A\}$. We see $|L| \geq 2$ since G is 3-connected. Let M^* be a

connected spanning subgraph of R such that $d_{M^*}(v) = \delta - 2$ for every $v \in A$. Note that for every connected spanning subgraph N^* of M^*, we have $d_{N^*}(w) = \delta - 2$ for $w \in L$ by the maximality of R. So every edge incident with w in M^*, where $w \in L$, is a cut-edge of M^*. Let $|L| = l$ and $|E(R) - E(M^*)| = m$. Then $l + m \leq t \leq \delta - 1$. We have

$$\omega(M^* - L) = \omega(H - L - (E(R) - E(M^*))) - 1 \geq (\delta - 3)l + 1.$$

So $\omega(H - L) \geq (\delta - 3)l + 2 - m$, which implies

$$\begin{aligned}\omega(G - u - L) &\geq (\delta - 3)l + 1 - m + 1\\ &\geq (\delta - 3)l + 2 - (\delta - 1 - l)\\ &= (\delta - 2)l + 3 - \delta.\end{aligned}$$

Since G is 3-connected, then

$$3\omega(G - u - L) \leq (\delta - 1)l + \delta - l.$$

It follows that

$$\omega(G - u - L) \leq \lfloor \frac{(\delta - 1)l + \delta - l}{3} \rfloor.$$

However

$$(\delta - 2)l + 3 - \delta - \frac{(\delta - 1)l + \delta - l}{3} = \frac{2\delta l}{3} - \frac{4\delta}{3} - \frac{4l}{3} + 3 > 0,$$

a contradiction. So we complete the claim and thus obtain a connected spanning subgraph T of H.

Let E' denote the set of corresponding edges of $E(T)$ in G. Then we obtain a spanning subgraph $T^* = \bigcup_{i=1}^{s} C_i \cup N^\delta(u) \cup E'$ of $G - u$ such that $d_{T^*}(v) \leq \delta - 2$ for every $v \in N^\delta(u)$. Thus the proof is complete. □

The following theorem is the main result of this section.

Theorem 3.2 *Let $G = (U, W)$ be a nice bipartite graph. If G is 3-connected, then G admits a vertex-coloring 2-edge-weighting.*

Proof. If G is a regular graph, the result follows from Lemma 2.4 (3). In the following, let G be a 3-connected non-regular bipartite graph with bipartition (U, W). Let $u \in U$ with $d(u) = \delta(G)$ and $N^\delta(u) = \{v \mid d(v) = \delta, v \in N_G(u)\} = \{u_1, \ldots, u_t\}$, where $t \leq \delta - 1$. Then by Lemma 3.1, there exist $e_1, \ldots, e_t$, where e_i is incident to vertex u_i in $G - u$ for $i = 1, \ldots, t$, such that $G - u - \{e_1, \ldots, e_t\}$ is connected.

By Lemma 2.4, we can assume that $|U||W|$ is odd. Now we consider two cases.

Case 1. $\delta(G)$ is even.

Then $|N(u) \cup (U - u)|$ is even. By Theorem 2.3, $G - u - \{e_1, \ldots, e_t\}$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in N(u) \cup (U - u)$ and $c(y)$ is even for all $y \in W - N(u)$. We assign every edge of $\{e_1, \ldots, e_t\}$ with weight 2 and every edge of $\{uu_i \mid i = 1, \ldots, t\}$ with weight 1. If $d(u_i) = d(u)$ for $i = 1, \ldots, t$, then $d_{G-u-\{e_1,\ldots,e_t\}}(u_i)$ is even. Now $c(u) = d(u)$ and $c(u) < c(u_i)$ for $i = 1, \ldots, t$. Moreover, $c(u_i)$ is even for $i = 1, \ldots, t$. Hence we obtain a vertex-coloring 2-edge-weighting of G.

Case 2. $\delta(G)$ is odd.

Then $|W - N(u)|$ is even. By Theorem 2.3, $G - u - \{e_1, \ldots, e_t\}$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is even for all $x \in N(u) \cup (U - u)$ and $c(y)$ is odd for all $y \in W - N(u)$. We again assign every edge of $\{e_1, \ldots, e_t\}$ with weight 2 and every edge of $\{uu_i \mid i = 1, \ldots, t\}$ with weight 1. Then $c(u) = d(u)$ and $c(u) < c(u_i)$ for $i = 1, \ldots, t$. Moreover, $c(u_i)$ is odd for $i = 1, \ldots, t$. Then we obtain a vertex-coloring 2-edge-weighting of G.

We complete the proof. $\square$

Based on the proof of Theorem 3.2, we can easily obtain the following corollary.

Corollary 3.3 *Let $G = (U, W)$ be a bipartite graph with $\delta(G) \geq 3$. If there exists a vertex of degree $\delta(G)$ such that $G - u - N(u)$ is connected, then G admits a vertex-coloring* 2*-edge-weighting.*

4 Conclusions

In this paper, we prove that every 3-connected bipartite graph has a vertex-coloring 2-edge-weighting. There exists a family of infinite bipartite graphs (e.g., the generalized θ-graphs) which is 2-connected and has a vertex-coloring 3-edge-weighting but not a vertex-coloring 2-edge-weighting. It remains an open problem to classify all 2-connected bipartite graphs admitting a vertex-coloring 2-edge-weighting.

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